Musical Harmony, Mathematics, and Esotericism

Celeste Jamerson and John F. Nash

Angels Playing Music. Detail from the Ghent Altarpiece by Jan and Hubert Van Eyck[1]

Summary

This article examines the historical development of musical harmony and explores its mathematical and esoteric underpinnings. Strong connections among music, mathematics and esotericism were recognized in the ancient world, famously by Pythagoras, and continue to be studied today. Although intervals in pitch are no longer defined by integer ratios; musical intervals, scales, and chords are still defined by mathematical relationships.

Increasing numbers of people are becoming aware of the esoteric significance of musical harmony, along with the esoteric dimensions of composition, performance and audition. The growing influence of the Fourth Ray, expected to usher in a new golden age of the arts and greater involvement of the Deva Evolution, will further enhance music's importance and open up new opportunities for service.

Introduction

When man emerged from the animal kingdom by the process of individualization, no doubt he brought with him his ancestors' repertory of sounds to attract mates, delineate territory, build community, and repel enemies. As his emotional and nascent mental facilities evolved, that repertory expanded to include what we could consider music. Singing, along with the use of drums and rudimentary wind and stringed instruments, developed at an early stage in human history.

About the Authors

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A flute, estimated to be 40,000 years old, was discovered in 2008 at a Stone Age site in southern Germany.[2] The discovery confirms that instrumental music extends back to the earliest migration of anatomically modern human beings into Europe. More importantly, it tells us that the music of the period was relatively sophisticated; the flute's five holes indicate awareness of a musical scale. By the third millennium BCE, both wind and string instruments were in use from Egypt and Mesopotamia to India and China. A musical score dating from the mid-second century BCE, and probably written for a lyre, has been discovered in Syria.

Both traditional religious teachings and the ageless wisdom assert that the universe was created by sound: "In the beginning . . . God *said*, Let there be light: and there was light."[3] The *Book of Job* records that "the morning stars sang together,"[4] and many cultures saw a connection between planetary motions and music. From very early times, music had a religious dimension. The world's oldest scriptural text, the *Rig Veda*, includes Sanskrit hymns, and the later *Samaveda* offers a much larger collection. Krishna is represented in Hindu artwork playing a flute. Both Apollo and Orpheus played lyres in Greek mythology, as did King David in Judaic tradition. Jewish temple chants evolved into the Byzantine, Ambrosian, Gregorian, and other chants of Christianity.

The classical study of musical intervals customarily is attributed to the Greek philosopher Pythagoras of Samos (c.570–c.495 BCE). Many of the underlying concepts were known earlier in Egypt, India and China,[5] but Pythagoras seems to have had profound insights into the nature and structure of music. Sadly, he left no written records, and we depend for knowledge of his achievements on the testimony of later writers. Pythagoras' work was recorded, interpreted and embellished by Plato (c.427–c.348 BCE), Nicomachus (c.60–c.120 CE), and many others. The Syrian Neoplatonist Iamblichus (c.245–c.325 CE), one of several who wrote biographies of Pythagoras, believed he was a god. Pythagorean harmonics was a school of music, mathematics and philosophy that evolved over a period of more than a millennium.

Pythagorean harmonics was built upon the recognition that certain musical intervals were more aesthetically pleasing than others, and that the favored intervals correlated with numerical ratios and geometric shapes. The same correlations resonated with, or were encoded into, the proportions of Greek temples and Gothic cathedrals. Those musical intervals and proportions were more than mere human conventions; they were believed to be part of divine revelation.

According to the intent of composers and performers, and according to its inherent quality, music can either stimulate or calm emotion. Importantly, it can send the mind soaring to high states of consciousness. Esoteric teachings recognize the effects of sound and music on human consciousness. The teachings also recognize music's ability to sweep devic beings into action. An order of music devas, the *Gandharvas*, is mentioned in Hindu and trans-Himalayan teachings. The Pseudo-Dionysius spoke of "choirs of angels," and Christian art frequently depicts angels playing musical instruments, a famous example being the fifteenth-century Ghent altarpiece by Jan and Hubert van Eyck, shown at the beginning of this article.

The topic of musical harmony, mathematics and esotericism is a vast one, and the present article is intended to explore some basic relationships. The first section, following this introduction, summarizes modern western music theory to establish the framework of terminology and concepts needed for the subsequent discussions in this article and forthcoming articles.

The second section is devoted to Pythagorean harmonics and the evolution of tuning systems in which musical intervals were defined by integer frequency ratios. In turn it discusses the eventual abandonment of such systems, because of their inherent weaknesses, and the acceptance of equal temperament, which is now the standard in western music. The third section explores the mathematical and esoteric associations of Pythagorean tuning and its derivatives, and notions of the harmony (or music) of the spheres.

A fourth section briefly explores modern esoteric teachings as they enhance our understanding of musical harmony. It also identifies the role of music in service. These various topics will be explored in greater detail in future articles that will focus on the relationship between music and color, and on music and the work of the Deva Evolution. The concluding section summarizes what has been learned herein and points to areas where further research is in progress or needed.

Mathematics plays an essential role in the discussion of music and esotericism. To achieve its objectives, this article makes use of mathematical concepts covered in typical high-school curricula. Today, mathematics has come to be regarded as a purely utilitarian discipline. Worse, our culture applauds otherwise-intelligent people who profess total ignorance of mathematics. We should remember that mathematics was once considered part of divine revelation and was taught in the mystery schools. Sacred mathematics remains an important area of esoteric philosophy.

Music, and the arts in general, are the special province of the Fourth Ray Ashram, headed by the Master Serapis. "At present," we are told, "He is giving most of His time and attention to the work of the deva, or angel evolution, until their agency helps to make possible the great revelation in the world of music and painting which lies immediately ahead."[6] All devas, we understand, are swept into activity by sound, but the Gandharvas' special mission is to work through music.

The great composers and musicians are believed to have been—or are now—senior disciples or initiates in the Fourth Ray ashram.[7] Opportunities on a large scale exist for aspirants and disciples to participate in the ashram's work, even just by simply listening to music of appropriate kinds and allowing it to speak to them on a soul level.

Sound, Pitch and Tuning

Sound consists of longitudinal waves of compression and rarefaction propagating through the air from a sound-producing source. A close analogy is the propagation of ripples on the surface of a pond. A musical sound—as distinct, say, from the sound of a pounding hammer—has a definite frequency, or *pitch*, and the length of the sound wave is inversely proportional to that frequency. High notes have higher frequencies and shorter wavelengths; low notes have lower frequencies and longer wavelengths.[8]

In 1955 the International Standards Organization defined the note A above middle C—conventionally referred to as A4—to be a sound wave of 440 cycles per second, or 440 Hertz (Hz).[9] This standard is referred to as concert pitch in Western music. A tuning fork at the pitch of A 440 is used as a reference to determine pitch and to tune instruments. A sound of 440 Hz has a wavelength of about 30 inches, or 76 cm. A3, one octave lower, has a frequency of 220 Hz and a wavelength of 60 inches. A5, one octave higher than A4, has a frequency of 880 Hz and a wavelength of 15 inches.

A standard has utilitarian value, but the choice of frequency has changed over time and is still not universally accepted. Baroque pitch varied considerably, depending on geographic and other factors. But generally, it was lower than modern concert pitch, and today's musicians playing Baroque music often choose a standard of A4 = 415 Hz. Scientific pitch, proposed in 1713 by French physicist Joseph Sauveur and briefly favored by composer Giuseppe Verdi, assigned a frequency of 256, or 2^8, Hz to middle C (C4), whereupon A4 acquired a frequency of 430.54 Hz.[10] Some esotericists, musicians, and even acoustic scientists claim that A4 = 432 Hz is more harmonious with the natural order, resonates with the heart chakra, and has a variety of other desirable properties.

Most people cannot hear frequencies lower than about 20 Hz, or above 20,000 Hz (20 kHz). The corresponding wavelengths are about 55 feet (16.7 meters) and 0.66 inches (1.67 cm), respectively. Music normally is composed of sounds within that audible range: equivalent to ten octaves.[11] A standard piano has a range of seven octaves.

Sound Production

Air can be induced to oscillate in a device with no moving parts. For example, it vibrates when blown over the lip of a flute. More commonly the oscillation is triggered by a vibrating object, such as a singer's vocal cords, a trumpet player's lips, the reed in a clarinet, a violin string, or the cone of an audio speaker.

In many musical instruments, the sound waves are amplified or modulated by internal resonance. For example, a standing, or stationary, wave forms in an organ pipe. If the pipe is open at one end, the longest sustainable wave is twice the length of the pipe; a wave of that length resonates and is amplified, while most other wavelengths are suppressed. An organ pipe is tuned to produce a particular pitch; so multiple pipes are needed to produce a range of notes. The longest pipe may be 64 feet (roughly 20 meters), while the shortest may be on the order of one inch (2.5 centimeters).

A recorder consists of a single tube, but it can produce a range of notes, by opening or closing air holes with the finger tips to alter its effective length. In other wind instruments the active length is altered by keys or valves. In a trombone the length of the tube is physically extended or shortened.

A vibrating string emits sound whose longest wavelength is the length of the string.[12] Pianos have many strings, each tuned to a particular note. A violin has only four strings, but their lengths can be shortened to produce higher notes, and the range of four octaves can be achieved by a skillful performer. String players change the pitch of their instruments by stopping off sections of the strings with one hand, while bowing or plucking the strings with the other.

In voice, the pitch is determined by the frequency of the opening and the closing of the vocal cords (more correctly named vocal folds, as this is their true shape), allowing air to pass between the folds to create sound waves of the desired frequency. At the pitch of A4, the vocal folds would be opening and closing 440 times per second. Needless to say, the frequency of these muscular movements is performed in response to signals from the brain and the nervous system falling mostly beneath the level of consciousness, due to the speed at which they are performed.

A tuning fork is designed to produce sound of a single frequency. When a note is sounded on a musical instrument, however, or when someone sings a note, the result is a richer sound consisting of a *fundamental frequency* and a set of higher frequencies referred to as *overtones*.

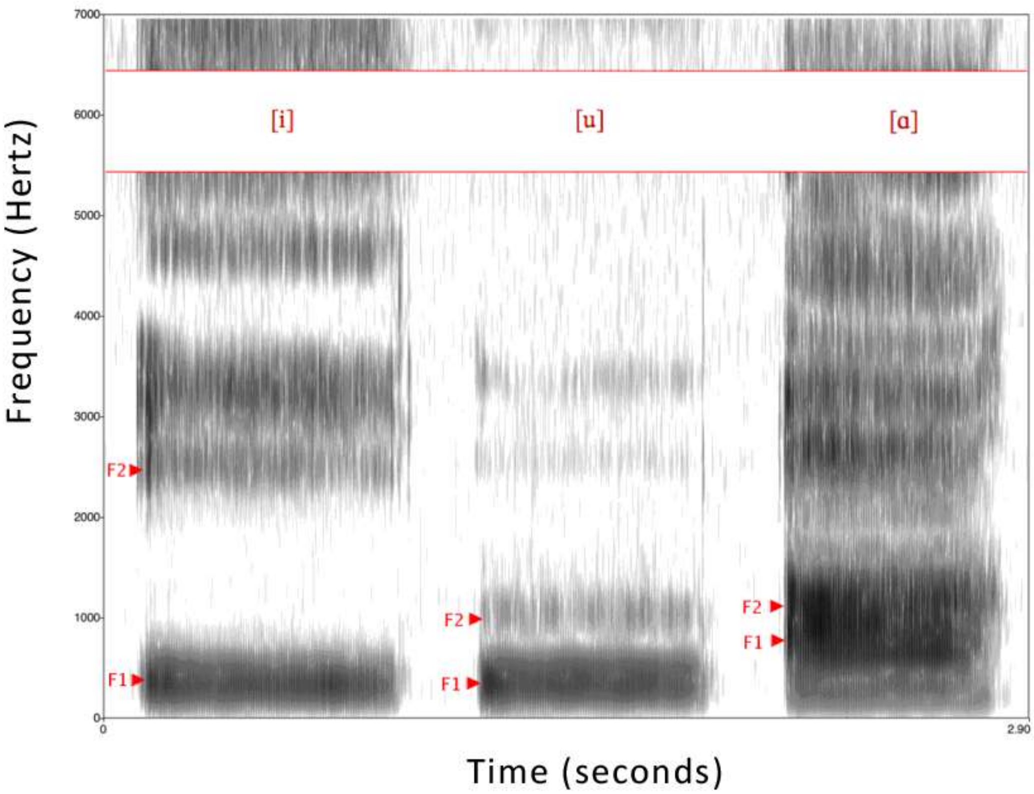
The fundamental frequency and its integer multiples: 1, 2, 3, 4 ..., make up the *harmonic series* based on the given fundamental. The first harmonic is the fundamental itself; the second harmonic is twice the fundamental frequency; the third harmonic is three times the fundamental frequency; and so forth. The harmonic series will be discussed in more detail later.

The fundamental, or first harmonic, is the lowest frequency that can be produced under the particular conditions. It is determined, for example, by the length of an organ pipe or piano string. Most types of musical instruments are designed to produce overtones that belong to the harmonic series. But percussion instruments—and defective instruments of other kinds—can produce *inharmonic* overtones: frequencies that are not integer multiples of the fundamental. Bells are particularly rich in inharmonic overtones.

Conventionally, overtones—that is, *harmonic* overtones—are numbered beginning with the second harmonic: the first *overtone* is the second harmonic; the second overtone is the third harmonic, and so on. Whereas the harmonic series—a theoretical construct—is infinite, a musical instrument or voice produces a limited number of perceptible overtones. The higher overtones tend to be weaker than the lower, but certain overtones may be particularly strong, and their relative strengths give an instrument or voice its characteristic sound quality, tone color, or timbre. They also help the listener identify differences among individual voices or even among instruments of the same type, such as violins of different instrument makers.

Regions of relatively strong harmonics are known as *formants*. In human speech and singing, they determine the vowels being heard. The strength of formants and harmonics can be plotted on a spectrogram. Figure 1 shows a spectrogram of the vowels [i] (as in "meet'), [u] (as in "moon") and [ɑ] (as in "father"). The brackets indicate that these are symbols in the International Phonetic Alphabet, or IPA. The vertical axis indicates pitch or frequency in Hertz. The horizontal axis indicates duration in seconds of the vowels as pronounced by the speaker. Darkness indicates amplitude or volume. The first formant is indicated by F1 and the second, which is higher, is indicated by F2. We can see, for example, that while [i] and [u] both have a relatively low first formant, the second formant of [i] is much higher than that of [u], giving it a brighter sound.

Figure 1. Spectral Analysis of the Vowels [i], [u] and [ɑ][13]



Sound waves can also be amplified or modulated by resonance within the performance space. The medieval cathedrals and college chapels were designed, not only with the liturgy and religious symbolism in mind, but also to help amplify and sustain the sounds of singers and musical instruments. *Reverberation*, the time taken for a sound to die away after its creation, is an important acoustic factor in large spaces; reverberation times in large cathedrals can be as long as five seconds. Reverberation is a serious problem for speech, forcing preachers to pause after every few words to allow the sound waves to decay. But it was used to great effect by choirs and instrumentalists in the Renaissance and Baroque periods. Italian composer Claudio Monteverdi (1567–1643) famously wrote music to exploit the acoustics of St Mark's Cathedral, Venice.

The sound produced by musical instruments or by gifted singers, in an appropriate performance space, is assumed to have aesthetic value: it is pleasing to the listener. That assumption lies at the heart of *music*, as distinct from *noise*, which might be created by a jackhammer or tornado. Ultimately, though, the distinction between music and noise is subjective and cultural; even within a culture people may disagree in their judgments.

Notes, Intervals and Scales

The basic building block of music is the *note*. A note has several characteristics, including pitch, duration, volume ("loudness"), and timbre.

In modern western music, notes are discrete frequencies that lie on a *scale*. The seven white notes on a piano are designated A through G, whereupon the next octave begins with another A, and the sequence is repeated. The five black notes produce sharps (♯) or flats (♭). The black note between A and B, for instance, can be viewed as either A♯ or B♭. There is no black note between B and C, or between E and F. Western music offers a basic seven-note, *diatonic*, scale, which, through the use of sharps and flats, can be expanded to a twelve-note, *chromatic*, scale.

The relation between two notes is referred to as an *interval*. Within an octave, intervals range from the unison, which is the same note, to the octave (eighth). Larger intervals are possible as well, such as the ninth, which is an octave plus a second. In figuring intervals, both the upper and the lower notes are counted, so the interval between adjacent notes is generally referred to as a second, for example. Important intervals, with respect to our later discussion, are the third, fourth, fifth and octave. The fourth and fifth are inversions of each other. For example, the interval between C and G is a fifth, while that between G and the next higher C is a fourth. Intervals may be *melodic* ("horizontal"): one note is played after the other, or *harmonic* ("vertical"): the notes are played simultaneously, one on top of the other.[14]

In modern western music the intervals between adjacent notes of the scale are either *whole tones* (whole steps) or semitones (half steps).[15] For example, the interval between A and B, or F and G, is a whole tone, while the interval between B and C, or between E and F, is a semitone. No black note appears between B and C, or between E and F, because there is no room for it in the progression of frequencies. Some contemporary compositions contain *microtones*, but western music generally does not admit intervals less than a semitone.

Every octave, as we move up the musical scale, represents a doubling in frequency. The assignment of frequencies to notes within the octave is referred to as *tuning*. Various tuning systems have been used in the past, and some of those will be discussed later. The system in general use today is called *equal temperament*. It assigns an equal frequency ratio to each of the twelve semitones in the octave: a ratio of $2^{1/12}$, or approximately 1.059.[16] A whole tone is two semitones, so its frequency ratio is $2^{1/12} \times 2^{1/12} = 2^{1/6}$, or approximately 1.122.[17]

A fourth is equivalent to a frequency ratio of $2^{5/12}$, or approximately 1.335, and the fifth a ratio of $2^{7/12}$, or approximately 1.498. The two ratios are very close to 4/3 and 3/2, respectively. We shall see later that the fourth and fifth were once *defined* by those frequency ratios.

Musicians and music theorists prefer to focus on the exponents (the powers of 2) rather than the ratios. They multiply the exponent by 1,200 to produce a value in *cents*. A semitone becomes $1/12 \times 1200 = 100$ cents; a whole tone becomes 200 cents, a fourth 500 cents, a fifth 700 cents, and an octave 1,200 cents. Most people can judge aurally a difference of twenty-five cents between two consecutive notes, and a few—like good piano tuners—can discern differences of five cents. A small difference between notes played simultaneously can easily be discerned because of the *beat* created; for instance, pitches of 440 Hz and 480 Hz produce a beat of $480 - 440 = 40$ Hz. The creation of unpleasant beats is a major reason why close musical intervals, like a second, sound discordant.

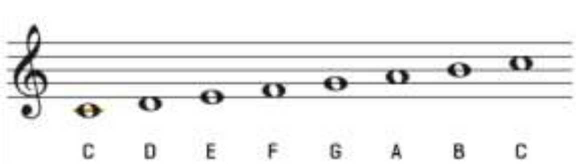
Converting from frequency ratios to cents involves a logarithmic transformation,[18] but the linearity of the values in cents—or, for that matter, the linearity of a piano keyboard—should not obscure the underlying reality that musical intervals are determined by frequency *ratios*. Nor should the even hundreds: 100, 200, and so forth, convey the impression that equal-temperament tuning is *a priori* the "correct" one. It is purely a convention, and the relative strengths and weaknesses of different tuning systems will be discussed later in the article.

Apart from the fourth, fifth and octave, which are normally "perfect," unless raised (augmented) or lowered (diminished), intervals are usually characterized as major or minor. For example, a major third extends over two whole tones, or 400 cents;[19] a minor third extends over one whole tone plus a semitone, or 300 cents. A major seventh corresponds to 1,100 cents, and a minor seventh to 1,000 cents.

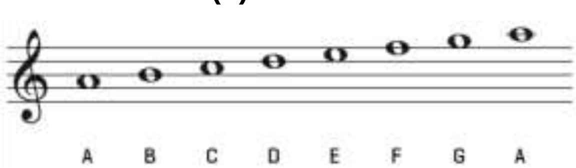
The major scale can be represented by the white notes on a piano, starting with C. The *natural* minor scale—as distinct from its variants the harmonic and melodic minor scales—can be represented by the white notes beginning with A (Figure 2).[20] The major and natural minor scales each contain two semitones, but they occupy different positions within the octave. In a major scale the semitones lie between the third and fourth notes (E and F), and between the seventh and eighth notes (B and C). In the natural minor scale they lie between the second and third notes (B and C), and between the fifth and sixth notes (E and F).

Figure 2. Major and Natural Minor Scales

(a) C Major



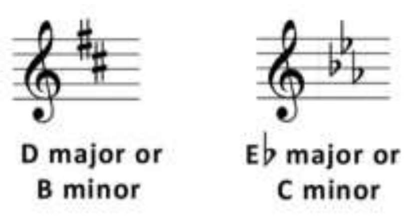
(b) A Minor



One does not always play the major scale starting on C, however, or the minor scale starting on A. A composer may choose a different key, or a performer may transpose an existing composition. The beginning note of a scale is called the *tonic*. Conceivably, it could be any note—white or black—on the keyboard, though certain choices are more common than others. For a given tonic, *sharps* or *flats* are introduced to preserve the proper pattern of whole tones and semitones in the corresponding scale. For example, a G-major scale requires F♯ to provide the semitone between the seventh and eighth notes of the scale. F♯ is the black note between the F and G white keys on a piano. The C-minor scale requires three flats: A♭, B♭ and E♭: the black notes between G and A, A and B, and D and E, respectively.

A key signature identifies sharps or flats to be observed throughout the piece, or until a new key signature is provided. Two examples are shown in Figure 3, one involving sharps and the other flats. It will be shown later that key signatures form a pattern illustrated by the Circle of Fifths. The key signature defines the key in which the piece or section is set. A pitch may be raised or lowered by the insertion of "accidentals," which are observed only in the measure, or bar, in which they occur. Accidentals provide compositional variety by temporarily departing from the prevailing key signature. They may take the form of a sharp, flat, or a *natural* (♮) which negates a previously established sharp or flat.[21]

Figure 3. Key Signatures



The major and minor scales are just two of the seven possible scales that can be generated by playing a sequence of adjacent white notes on a piano. For instance, the Phrygian mode can be represented by playing the white notes starting on E; its semitones lie between the first two notes and between the fifth and sixth notes of the scale. Ancient philosopher–musicians such as Pythagoras and Plato heard in the various modes distinctive qualities: amorous, warlike, lethargic, energetic, and so forth. Medieval church music could also be in different modes, but western classical music eventually settled on the major and minor modes, or scales, as the most pleasing and essentially abandoned use of the other modes.

Music theorists speak of scale *degrees*, which are notes measured relative to the tonic, regardless of the scale's key. The first degree is the tonic itself, and the second through seventh degrees are successive intervals above the tonic. The degrees have names, some of the most common being the *mediant*, the third degree; the *subdominant*, the fourth degree; and the *dominant*, the fifth degree.

Notes can also be designated, regardless of key, by the pedagogical system of *solfège*, or *solfeggio*. For example, the major scale is designated by the syllables: Do, Re, Mi, Fa, Sol, La, Ti (or Si), and finally Do in the next octave. Some authorities trace the solfège system back to the eighth-century Gregorian Chant *Ut queant laxis*, or "Hymn to St John the Baptist."[22] Two versions of the solfège system are in use. In the Movable Do system, common in the United States, "Do" designates the first note of the scale in any major key, with the other syllables following in succession. In the Fixed Do system, Do always refers to the note C, regardless of the key; Re always refers to the note D, Mi to E, and so forth.

Harmonic Series

Harmonics are integer multiples: 1, 2, 3, ... of the fundamental frequency. Harmonics form a mathematical pattern known as the *harmonic series*. The first sixteen harmonics, beginning two octaves below middle C, are shown in Figure 4. By convention the fundamental pitch is referred to as the first harmonic. The second, fourth, eighth, and sixteenth harmonics are successive octaves of the fundamental. Notes one or more octaves apart resonate closely with one another. We speak of *octave equivalency*, which means that all As are, in a sense, the same note; all Bs are the same; and so forth.

Figure 4. Harmonic Series Based on C

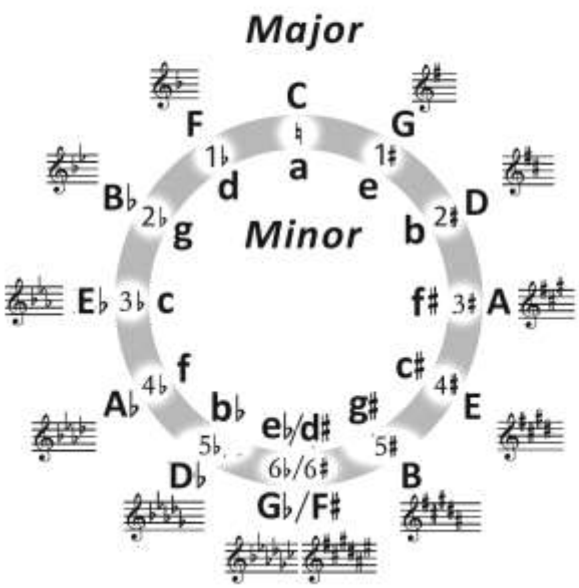
Other harmonics lie within the octaves, and the pitches get progressively closer together as the series progresses. Some of them closely approximate notes on the modern musical scale. For example, the third harmonic—a major fifth above the second harmonic—is very close to G below middle C in the modern musical scale. By contrast the seventh, eleventh and fourteenth harmonics only roughly approximate notes on the modern musical scale and would sound noticeably discordant if played together with their nearest piano notes.[23]

The relationship between the harmonic series and musical scales is an issue to be discussed later in this article. Suffice it to say, at this point, that the modern western musical scales—major and minor—resulted from a long process of evolution and represent a compromise among conflicting aesthetic, practical, and other factors. In times past people felt strongly that other musical scales were superior, or even had divine sanction. The modern scales have imperfections, but they allow multiple instruments to play together without excessive discordance, and they allow compositions to be transposed to different keys without excessive distortion.

Circle of Fifths

The interval of a fifth is of the greatest importance in music theory, and successive transpositions of a fifth reveal an interesting pattern in the key signatures (Figure 5).[24] This pattern is known as the Circle of Fifths.

Figure 5. Circle of Fifths Showing the Key Signatures of their Respective Major and Minor Scales.



The C-major scale requires no sharps or flats; it can be played on the white notes of a piano. G is a fifth above C, and the G-major scale requires one sharp (F♯). D is a fifth above G, and the D-major scale requires two sharps (F♯ and C♯). Further transpositions add more sharps, until one reaches the scale of F♯ major, which requires six sharps (F♯, C♯, G♯, D♯, A♯ and E♯). Moving in the opposite direction, F is a fifth below C, and an F-major scale requires one flat (B♭). B♭ is a fifth below F, and a B♭-major scale requires two flats (B♭ and E♭). Again, successive transpositions add more flats, until one reaches G♭ major, which requires six flats (B♭, E♭, A♭, D♭, G♭ and C♭). F♯ and G♭ are, of course, the same note, and the F♯-major scale and the G♭-major scale are played with the same notes on the piano. Twelve major scales can be constructed, each beginning on a particular white or black note on a piano, within the span of an octave. These scales and their respective key signatures are displayed around the outside of the circle shown in Figure 5.

Minor scales transposed by successive fifths yield a similar result, and these are shown on the inside of the circle in Figure 5 (to distinguish them in the figure, minor keys are designated by lower-case letters). The A-minor scale needs no sharps or flats, so A lies at the top of the circle. E is a fifth above A, and the E-minor scale requires an F♯. Correspondingly, D is a fifth below A, and the D-minor scale requires a B♭. Twelve minor scales can be constructed. The scale of D♯-minor, with six sharps—or equivalently E♭-minor, with six flats—lies at the bottom of the circle.

Chords

In the medieval church, Gregorian chant was sung in unison. Or, singers might sing an octave apart, invoking the principle of octave equivalency, in which notes of the same letter name are said to share the same fundamental quality.

In due course, liturgical chant began to incorporate *organum*, or a second line of harmony at an interval of a perfect fifth or a perfect fourth from the main pitch. It may be seen that these intervals correspond to the distance between the second and third harmonics and between the third and fourth harmonics of the harmonic series given above. Eventually, as tempered tuning became more common, the harmonies of western classical music began to incorporate the major and minor third.

Some instruments, like a flute, can only play one note at a time. Others, like a piano, can play chords, consisting of multiple notes, and combinations of instruments can obviously do the same. Chords introduce harmony to the composition to produce a pleasing—or sometimes disturbing—psychological effect.

Chords are most often based on intervals of thirds and fifths, or their respective inversions, the sixths and the fourths. The most basic chord is the triad, composed of three notes. The simplest triad consists of the *root*—the note on which the chord is built[25]—and the third and fifth above it—or equivalently, two successive thirds. A major triad consists of a minor third stacked top of a major third to create a perfect fifth between the top and bottom notes. In a minor triad, a major third is stacked on top of a minor third, also resulting in a perfect fifth between the top and bottom notes of the triad. A diminished triad consists of two minor thirds stacked on top of one another, creating an interval of a diminished fifth between the root and the fifth of the chord. In an augmented triad, two major thirds are stacked on top of one another to create an interval of an augmented fifth between the bottom and top notes (Figure 6).

Figure 6. Triads Built on the Root of C

In practice, some of the notes of the triad are often doubled, especially in music written in four-part choral or instrumental textures. Compositions commonly end on a tonic triad with the root in the bottom and top voices. Psychologically, this arrangement has a satisfying sense of finality. By contrast, psychological tension or anticipation can be evoked by slight modification. For example, if an interval of a seventh above the root is included in the chord, a dissonance results, and the beat between the seventh and the root's second harmonic creates a sense of incompleteness, as the listener waits for the chord to be resolved to a more consonant harmony.

Chords can be built on any degree of the scale, and each has its own tonal function in western music. A chord built on the tonic is known, appropriately, as the *tonic* chord and serves as a home base for pieces written in that key. Usually, a piece will begin and end on a tonic triad. The dominant chord, built on the fifth note of the scale, tends to be the second most important in a composition and provides an important "pull" against the tonic. The subdominant chord, built on the fourth note, can serve as a bridge between the tonic and the dominant chords. The chords built on the other notes on the scale also serve important functions.

Musical Scales Based on Integer Ratios

For many centuries the intervals in musical scales were defined, not by fractional powers of 2, as in modern equal temperament, but by the ratios of *integers*, or whole numbers. Moreover, it was believed that the integers should be small, though that ideal was hard to achieve. Pythagoras created a tuning system in which the ratios were restricted to integer powers of 2 and 3, such as 3/2, 4/3 and 9/8. Later tuning systems admitted ratios such as 5/4 and 5/3.

Pythagorean Concepts of Music

Pythagoras famously experimented with a monochord (Figure 7) to investigate the pitch of tones generated by vibrating strings. A moveable bridge isolated a portion of the stretched string to change its active length and corresponding pitch. The original monochord may have had, as its name implies, a single string, but a fifteenth-century CE manuscript shows Pythagoras using an instrument with six strings.[26] Multiple strings enable the user to sound both melodic and harmonic intervals.

Figure 7. Monochord with Three Strings[27]

Since the bridge could be set at any arbitrary position along the string, an unlimited number of intervals could be created. But Pythagoras identified those that seemed the most aesthetically pleasing. They represented Order, emerging from the Chaos of infinite possibility. Reducing the active length of the string by one-half raised the pitch by an octave, reducing it to one-quarter of its original length raised the pitch by two octaves. The octave resonated closely with the original note; in some sense it was the same note.

Two other significant intervals were produced when the vibrating string was two-thirds and three-fourths of its original length. We know those intervals as the perfect fifth and perfect fourth, respectively. Small numbers evidently produced the purest sounds, and Pythagoras' favored sounds were produced when the monochord string was divided into fractions formed from the first three natural numbers: 1, 2, 3, together with multiples like $2 \times 2 = 4$, $3 \times 3 = 9$.

Pythagoras lacked the technology to measure frequency, but he correctly inferred that it varied inversely with string length. The frequency ratios corresponding to the fourth, fifth and octave— obtained when the string length was divided in the ratios 3/4, 2/3, and 1/2, respectively—were 4/3, 3/2. and 2/1 (or simply 2).

The tonic, fourth, fifth and octave provided a rudimentary musical scale. The frequency ratios: 1, 4/3, 3/2 and 2, could be converted to integers by assigning the tonic a value of 6, whereupon the frequencies would be proportional to 6, 8, 9 and 12. Table 1 shows the scale in the key of C. C' denotes the octave above C.[28]

Table 1. Pythagorean Four-Note Scale

Note	**C**	**F**	**G**	**C'**
Frequency ratio	1	4/3	3/2	2
Frequency number	6	8	9	12

Pythagoras' rudimentary scale was too limited to produce meaningful music; four more notes were needed to produce a seven-note, diatonic, scale. To generate a C-major scale: D, E, A, and B must be created. Each note would correspond to a particular frequency ratio, and Pythagoras insisted that these ratios must be formed from the numbers: 1, 2 and 3, together with their powers: $2^2 = 4$, $2^3 = 8$, $3^2 = 9$, $3^3 = 27$, and so forth. Moreover, the numbers must be as small as possible to guarantee purity of sound.

The additional notes can be created in alternative ways. One method observes that the fourth (F) and fifth (G) are a whole tone apart, and their frequencies are in the ratio 9/8. So, working from C to D and E, and from G to A and B, successive whole tones can be created by multiplying the frequencies by 9/8. The resulting diatonic scale is shown in Table 2. Intervals in cents are shown for comparison with the modern musical scale. We recall that the whole tone in the modern scale is 200 cents, and the semitone is 100 cents.

Table 2. Pythagorean Diatonic Scale

Note	**C**	**D**	**E**	**F**	**G**	**A**	**B**	**C'**
Frequency ratio	1	9/8	$(9/8)^2$ = 81/64	4/3	3/2	3/2 x 9/8 = 27/16	3/2 x $(9/8)^2$ = 243/128	2
Frequency number	6	6.75	7.59375	8	9	10.125	11.390625	12
Interval in cents	-	204	408	498	702	906	1,110	1,200
Modern scale	-	200	400	500	700	900	1,100	1,200

The alternative method is to multiply the frequency of the tonic by successive powers of 3/2 to generate the series of fifths. For example, the fifth above C is G, with a frequency ratio of 3/2. A fifth above G is D'—that is the D above C'—with a frequency ratio of $(3/2)^2$, or 9/4. Continuation of the process yields the notes shown in Table 3. C" is two octaves above C, and E" and B" are the respective notes in the octave beginning with C". When the notes are mapped onto the reference octave, by dividing the ratios by appropriate powers of 2 (octave equivalency), the scale is identical to the one shown in Table 2.

Table 3. Successive Fifths

Frequency ratio	**Actual note**	**Divisor**	**Adjusted ratio**	**Note mapped to reference octave**
1	C	1	1	C
3/2	G	1	3/2	G
$(3/2)^2$ = 9/4	D'	2	9/8	D
$(3/2)^3$= 27/8	A'	2	27/16	A
$(3/2)^4$= 81/16	E"	4	81/64	E
$(3/2)^5$= 243/32	B"	4	243/128	B

The frequency ratios in the Pythagorean diatonic scale are still all powers of 2 and 3, but the original simplicity has been lost. In order to make the frequency numbers all integers, C would now have to be assigned a value of $2^7 \times 3 = 384$, instead of 6. The ideal of small integers could no longer be realized.

The interval between E and F corresponds to a frequency ratio of $2^8/3^5$ = 256/243, or approximately 1.0535. This interval was termed a *lemma*, literally "left over." An equal lemma occurs between B and C'. Such lemmas were later identified as semitones, but they are only 90 cents—smaller than the modern semitone of

100 cents. By contrast, the whole tones are 204 cents, slightly larger than the modern whole tone. The modern fourth and fifth are within two cents of their Pythagorean counterparts.

A complementary scale can be constructed by working backward from C' to B and A, and from F to E and D, successively multiplying the frequencies by 8/9; or, alternatively, by creating the series of descending fifths, starting with C'. The result resembles the natural minor scale, except that a semitone, or half step, lies between the first and second notes of the scale, rather than between the second and third. It contains the notes D♭, E♭, A♭ and B♭. The lemmas, or semitones, lie between C and D♭, and between G and A♭.

Pythagorean harmonics was built upon the recognition that certain musical intervals were more aesthetically pleasing than others, and that the favored intervals correlated with numerical ratios and geometric shapes. The same correlations resonated with, or were encoded into, the proportions of Greek temples and Gothic cathedrals. Those musical intervals and proportions were more than mere human conventions; they were believed to be part of divine revelation.

Combining the two scales yields eleven of the twelve notes needed to form a chromatic, scale. Only F♯, or G♭, in the very middle of the scale, is missing. The interval between the tonic and this note is referred to as the *tritone*. One way to find the tritone is to multiply the frequency of F, or divide that of G, by the Pythagorean *lemma*, of 256/243. Unfortunately, this method yields distinct pitches for F♯ and G♭; their frequencies are $3^{12}/2^{19}$, or about 23.5 cents, apart—well within the range of auditory discernment.[29] Although traditionally a very dissonant interval needing careful resolution, the tritone was featured thematically by Bernstein in *West Side Story* and by Benjamin Britten in his *War Requiem*.[30]

Another problem, acknowledged by Pythagoras himself, is that the series of fifths cannot generate the octave; no nonzero power of 3 can ever equal a power of 2. A "close encounter" is obtained by twelve fifths, which exceed seven octaves by a ratio of $3^{12}/2^{19}$, or 23.5 cents, precisely the same interval that plagued the tritone. It is referred to as the *Pythagorean*, or *diatonic*, *comma*.[31] The comma was of little practical concern, since musicians largely confined themselves to a single octave: a comfortable range for a human voice. The irreconcilability of the series of fifths and octaves was theoretically worrisome, however, and threatened belief in the divine endorsement of integer ratios in music.

Other problems were of a practical nature. The semitone, with a frequency ratio of 256/243, or an interval of 90 cents, was too small. And the major third of 408 cents was too large, causing the triad chord, CEG, to sound discordant. Composers and musicians steered clear of the major third until the Renaissance, when tempered scales began to appear.[32]

Evolution of the Musical Scale

While the Pythagorean scale was imbued with great mathematical and—as we shall see—esoteric significance, it had inherent weaknesses that prompted efforts to improve upon or replace it. Musicians were loathe to abandon the principle that small-integer frequency ratios produced the most pleasing intervals. A solution was to increase the range of acceptable integers beyond powers of 2 and 3.

As early as the second century CE, Claudius Ptolemy proposed that 5 be included. This was an important number since the fifth harmonic, transposed to the reference octave, creates the major third, with a ratio of 5/4. Ptolemy's proposal received little attention until Bartolomé Ramos de Pareja (c.1440 –1522) reintroduced it and offered what we now call *just intonation* tuning.[33] Ramos used the ratio 5/4 to generate the major thirds, from C to E, from F to A, and from G to B (Table 4).

Table 4. Just Intonation Tuning

Note	C	D	E	F	G	A	B	C'
Frequency ratio	1	9/8	5/4	4/3	3/2	4/3 x 5/4 = 5/3	3/2 x 5/4 = 15/8	2
Frequency number	6	6.75	7.5	8	9	10	11.25	12
Cents	-	204	386	498	702	884	1,088	1,200
Modern scale	-	200	400	500	700	900	1,100	1,200

Just intonation gave musical scales new life. Not only did it preserve the rule of small integers, some were smaller than those required in Pythagorean tuning. For example, the major sixth from C to the A above is a frequency ratio of 5/3, compared with the Pythagorean ratio of 27/16. The major seventh is 15/8, instead of 243/128. The frequency numbers in Table 4 can all be made into integers if the tonic is assigned a value of 24, compared with the 384 needed in the Pythagorean major scale.

Musicians welcomed just intonation tuning because of the improvement to the major third, and chords involving thirds and fifths, such as CDF, became esthetically feasible for the first time. Yet other chords were unsatisfactory. For example, the fifth from D to A has a frequency ratio of 40/27, or 680 cents, compared with the fifth from C to G, with a ratio of 3/2, or 702 cents. Transposition from one key to another remained problematic.

A century after Bartolomé Ramos, astronomer Johannes Kepler (1571–1630) conducted monochord experiments to extend the range of integer ratios still farther. He reproduced Pythagoras' results and confirmed the Ptolemy/Ramos frequency ratios of 5/4 for the major third and 5/3 for the major sixth. Kepler also proposed a ratio of 6/5 for the minor third—D♯ or E♭ on a C major scale—and 8/5 for the augmented fifth, or minor sixth: G♯ or A♭. But those pitches were too high—316 and 814 cents, respectively—reducing the semitones between them and the next-higher notes to only 70 cents. Kepler's contribution to tuning was minimal. We shall see, however, that he suggested interesting correspondences between musical scales and planetary motions.

The problems could have been overcome by incorporating ratios of larger and larger integers. But that ran counter to the core Pythagorean principle that pure intervals demanded small integers. Instead, by the seventeenth century, musicians were seeking more pragmatic solutions. One solution was to increase the number of pitches in an octave, to allow musicians to choose which version of a note to play. Keyboards were designed with as many as thirty-two keys to the octave.[34] Another solution was to *temper* offending ratios, essentially multiplying their frequency ratios by "fudge factors" to avoid dissonance. All forms of temperament met with strong opposition from classical purists. Preachers across Europe took to their pulpits to condemn musicians and musicologists for violating the divinely ordained principle of integer ratios.

The "final solution" was the equal temperament system described earlier in this article. It abandoned the reliance on integer ratios, assigning instead a frequency ratio of $2^{1/12}$, or 100 cents, to every semitone in the musical scale. Equal temperament even compromised the perfect fourth and fifth, albeit by only two cents. The use of fractional powers of 2 might seem like a modern innovation, but Greek mathematician Archytas (428–347 BCE) proposed that the major third be defined by the cube-root of 2, that is $2^{1/3}$ or precisely 400 cents.[35] Archytas' proposal anticipated equal temperament by more than two millennia.

Music, Mathematics and Esotericism

Pythagoras taught that arithmetic was "number in itself," music was "number in time" (or "music in motion"), geometry was "number in space," and astronomy was "number in time and space." The four topics became the *quadrivium* of classical education. The emphasis on numbers did not imply a fixation with counting; rather, numbers were believed to be divine entities. Numbers were for Pythagoras what *Forms* were for Plato, and Hebrew letters for the Kabbalists. They were aspects of divine revelation, and, with them, music, geometry, and the patterns of planetary motion.

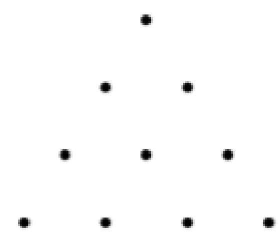
Pythagoras is said to have used music to bring about positive change in the lives of his pupils. Certain melodies were used to offset an excess of emotions or states of being such as anger, sorrow, lethargy or combativeness. Appropriate melodies were played upon going to sleep at night and awakening in the morning. Pythagoras was even said to be able to heal certain diseases of the body by means of music. This healing and uplifting power of music was connected to Pythagoras' understanding and use of the mathematical laws of music which enabled him to reproduce an earthly version of the "music of the spheres."[36] For him, music was a kind of sympathetic magic.

Mathematical and Esoteric Properties of Musical Scales

The numbers appearing in the Pythagorean musical scales are replete with mathematical and esoteric significance. The numbers 1, 2, 3, 4, which form the basis of the rudimentary four-note scale, are found in the geometric figure known as the *tetraktys* (Figure 8).

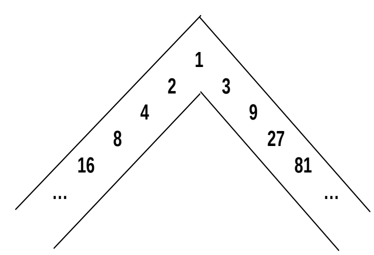
The frequency numbers: 6, 8, 9, 12, in the rudimentary musical scale, have the property that 8 is the harmonic mean of 6 and 12, and 9 is their arithmetic mean: $8 = 2 \times (6 \times 12)/(6 + 12)$, $9 = (6 + 12)/2$.[37] *Harmonic mean* is part of the standard mathematical vocabulary, but its origins clearly lie in musical analogies. The tritone in the equal-temperament musical scale is the geometric mean of 6 and 12, and also of 8 and 9: $\sqrt{(6 \times 12)} = \sqrt{(8 \times 9)} = 8.485...$, equivalent to precisely 600 cents.[38]

Figure 8. Tetraktys



All the frequency ratios in the Pythagorean diatonic scale are powers of 2 and 3 (even 1 is the *zeroth* power). They can be arranged in a geometric figure known as the Pythagorean Lambda—so named because of its resemblance to the Greek letter Λ (Figure 9). The powers of 2 are shown in the left-hand leg, and the powers of 3 in the right-hand leg.

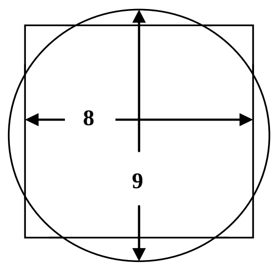
Figure 9. Pythagorean Lambda



The numbers 2 and 3 were believed to be the first emanations from the divine Unity. If 1 represented the unmanifest spirit, 2 represented its first manifestation, and 3 the resulting new creation. Odd numbers, which reached out into new realms of manifestation, were deemed to be masculine, while even numbers, which restored harmony and balance, were feminine. Two signified the first feminine energy and 3 the first masculine energy. The Pythagorean scale represented harmony between spirit and matter, heaven and earth, the masculine and the feminine.

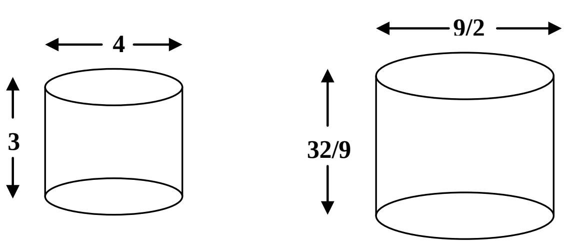
The Pythagorean whole tone has a frequency ratio of 9/8. This ratio was significant to the Greeks because of its relevance to the problem of “squaring the circle”: constructing a square with the same area as a given circle (Figure 10). We now know that the ratio of the diameter of the circle to the side of the square is $2/\pi^{1/2}$, or 1.12838… For comparison, 9/8 = 1.125, a difference of only 0.5 percent.

Figure 10. “Squaring the Circle”



A surprising geometric result, with relevance to the Pythagorean diatonic scale, was known to the ancient Egyptians. It involves two barrels. The smaller barrel has a height of 3 units and a diameter of 4 units; the larger one has a height of 32/9 and a diameter of 9/2 (Figure 11).

Figure 11. Two Barrels



The following ratios emerge, all with musical significance:

- Ratio of the diameter to the height of the small barrel = 4/3, the perfect fourth; for example, the interval between C and F, or between G and C'.
- Ratio of the surface areas of the two barrels = 4/3, again a perfect fourth.
- Ratio of the volumes of the two barrels = 3/2, a perfect fifth; for example, the interval between C and G.
- Ratio of the diameters of two barrels = 9/8, a whole tone; for example, the interval between F and G.
- Ratio of the diameter to the height of the large barrel = 81/64, a major third; for example, the interval between C and E.
- Ratio of the heights of the two barrels = 32/27, a minor third; for example, the interval between E and G.

Seven, the number of notes in the diatonic scale (excluding the octave), had great significance in ancient times. Importantly for Pythagoras, it was equal to 2 + 2 + 3, his favored integers. Seven combinations can be formed from three elements: if the elements are denoted by X, Y and Z, the possible combinations are: X, Y, Z, XY, XZ, YZ and XYZ. It was the virgin number, the number of creation. Seven, the *heptad*, was sacred to the planet Venus and the Goddess Athena. Orpheus’ lyre allegedly had seven strings. There were seven vowels in the Greek alphabet; seven days in the week; and seven planets, including the Sun and Moon.[39] Seven is the largest, single-digit prime number.

Plato greatly admired Pythagoras, and he embraced the notion that the musical scale was an analogy (*analogia*) of the created universe. In his dialogue *Timaeus*, Plato asserted that God created the “body of the universe” from the elements Fire and Earth. Normally “it is not possible to combine two things properly without a third to act as a bond to hold them together.”[40] In an interesting subtlety, however, this particular bond required *two* intervening elements, Air and Water, since the universe was “solid.” The four elements were related by the formula: “fire was to air as air to water, and air was to water as water to earth.”[41]

That formula can be interpreted mathematically as a pair of simultaneous equations, which can be solved to determine the corresponding musical intervals.[42] If Earth is assigned the value 1, and Fire 2, to delineate the octave, Water and Air turn out to be—not 4/3 and 3/2, the ratios of the perfect fourth and fifth, as one might have hoped—but $2^{1/3}$ and $2^{2/3}$, respectively. These latter values, equivalent to 400 and 800 cents, correspond precisely to the major third and minor sixth in the modern musical scale. Plato's major third is the same as Archytas,' mentioned earlier, while his minor sixth of 800 cents provides another ancient precedent for equal temperament.

The soul on its tiny scale can create "the new man" by the power also of sound, and a musical rhythm can usefully be imposed upon the personality life by the disciple. . . . Let love and light and music enter more definitely into your daily life. . . . give your mind the opportunity, through the massed sound of music, to break down the personality-imposed barriers between the free flow of soul life and you.

With the adoption of just intonation, the fundamental set of numbers was now 1, 2, 3, and 5. No longer could the numbers be equated to Unity and its first two manifestations: the female and the male; nor could the numbers be found from Pythagoras' Lambda. Nevertheless, 5 had important properties, even to Pythagoreans. It is the hypotenuse of a right-angled triangle whose other sides are 3 and 4, one of very few in which all the sides are integers. Furthermore, the area of the triangle is 6, the frequency number assigned to the tonic. For Aristotle, 5 signified "marriage."[43]

Extension of the basic number set still did not solve all the practical problems, and musicians turned to tempered musical scales. Temperament met not only with religious objections, but also with objections from the esoteric community. Fabre d'Olivet (1767-1825), esotericist and minor composer, wrote that equal temperament robbed music of its expressive and spiritual power. D'Olivet referred to the ancient Greeks who believed that music had miraculous power over the mind, body and emotions. It was plain that modern music did not possess this miraculous power. While some writers concluded that these claims must have been exaggerated, D'Olivet wrote that the loss of music's ability to work miracles was due to the adoption of the system of equal temperament and the ensuing distortion of the pure intervals used by Pythagoras and the ancients in their music.[44]

While equal temperament has made possible much of the great music in the classical western tradition, it is not universally popular. Some musicians claim that it is inappropriate for the performance of music of earlier periods and that it obscures the differences between the various major and minor keys, each of which is said to have its own character. Some purists still complain that the temperament of the Pythagorean fourth and fifth, albeit by only 2 cents, is not only detectable but toxic.

Now that equal temperament has become almost universal in the West, fewer opportunities arise to discover new arithmetic and geometric correspondences. But our pattern-seeking instincts have not gone away. Much has been made of the configuration of keys on the piano keyboard. For instance, it has been noticed that each octave includes one white note, D, between the two black notes C# and E♭ ; a series of three adjacent black notes: F#, A♭ and B♭; a total of five black notes; eight white notes; and thirteen notes in the chromatic scale (including the repeated tonic). The numbers: 1, 2, 3, 5, 8 and 13 are the first six numbers in the Fibonacci series, in which numerologists find profound meaning.[45]

Harmony of the Spheres

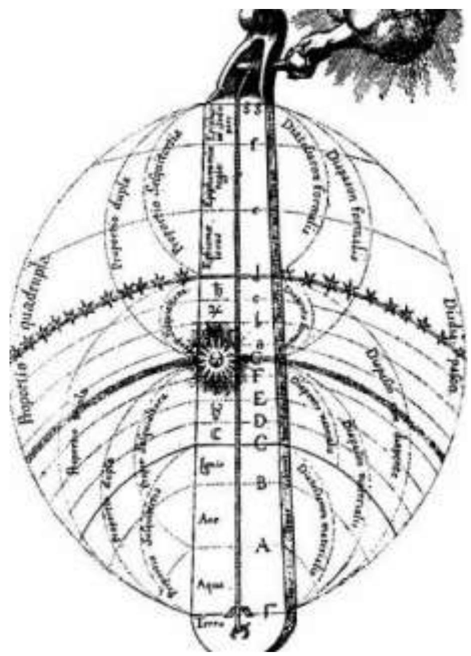
The music, or harmony, of the spheres fascinated many people in antiquity. It was said that Pythagoras could actually hear it, but most of his followers were content to interpret the celestial harmony as something mathematically elegant or satisfying. If—to revisit the quadrivium—music was "number in time," and astronomy was "number in time and space," it was reasonable to assume that the geometric motion of the planets created music.

Several attempts were made to associate the planets with notes on a musical scale, though the results were not always consistent. For example, Plato assigned tones to the planets in an ascending scale, reflecting their perceived distances from a stationary Earth. From the Moon to Saturn, the most distant planet known at the time, was a major seventh; the octave was assigned to the sphere of the fixed stars. Five centuries later, Nicomachus took the opposite view, assigning the planets to a descending scale, reflecting their relative velocities. He assigned a high note to the Moon, which moved fastest, and a low note to the slow Saturn; the immovable stars comprised the tonic.[46]

Pythagorean concepts of music passed into Neoplatonism and, in due course, into Hermeticism: an amalgam of astrology, alchemy, and ceremonial magic. Christian Hermeticist Cornelius Agrippa von Nettesheim (1486–1535) described a variety of magic rituals, each with appropriate words of power and planetary correspondences. Invocation of the planets could include musical tones and intervals. Agrippa related the Sun to the octave or double octave, Mercury to the perfect fourth, and Jupiter to the perfect fifth.[47]

Robert Fludd (1574–1637) conceived of a divine monochord extending from Earth to Heaven and spanning a double octave (Figure 12). The monochord, expressing the harmony of the spheres, was tuned by the very hand of God. The first through fourth musical intervals correspond to the four elements, Earth, Water, Air and Fire, respectively. The planets, Moon through Saturn, occupy the next seven intervals; the Sun is the octave, lying at the midpoint between Earth and Heaven. The fixed stars occupy the fifth of the second octave.[48] Fludd's cosmological model was still geocentric, even though the work of Nicolaus Copernicus and Galileo Galilei was already well-known.

Figure 12. Robert Fludd's Celestial Monochord



Fludd's contemporary, astronomer Johannes Kepler (1571–1630), carried music theory and its esoteric associations into the heliocentric age. And whereas Fludd and others approached the harmony of the spheres metaphorically, Kepler studied it analytically. Kepler's *Harmonices Mundi* (1619) covers much ground, but it includes his greatest contribution to astronomy: the discovery that planetary orbits are elliptic rather than circular—another blow to classical purists who affirmed the divinity of the circle.

Kepler discovered that, for certain planets, the ratios of their maximum and minimum angular velocities—at perihelion and aphelion, respectively—approximate harmonic ratios. The greater the eccentricity of the orbit, the greater the ratio. For instance, Mercury has a high eccentricity, and its ratio is 12/5, corresponding to an octave plus a minor third. By contrast, Venus's orbit is nearly circular, and its ratio is only 25/24—no more than a "comma." Ratios for the planets known in Kepler's time are listed in Table 5.[49]

Table 5. Kepler's Planetary Musical Intervals

Planet	Eccentricity	Velocity Ratio	Musical Interval	Choir Section
Venus	0.007	25/24	Comma	Alto
Earth	0.017	16/15	Semitone	Alto
Jupiter	0.048	6/5	Minor third	Bass
Saturn	0.054	5/4	Major third	Bass
Mars	0.093	3/2	Fifth	Tenor
Mercury	0.206	12/5	Octave + minor third	Soprano

A planet's velocity varies as it orbits the sun, and Kepler imagined the planets "singing" ascending and descending scales within their ranges. Mercury's range was more than one octave, whereas Earth's was a mere semitone, and Venus' a single note. Kepler suggested that the celestial choir consisted of a tenor (Mars), two basses (Saturn and Jupiter), a soprano (Mercury), and two altos (Venus and Earth).[50] More-recently discovered planets presumably have swelled the choir's ranks. Neptune (eccentricity 0.009) has joined the alto section; Pluto (0.249), is a new soprano; Uranus (0.047) is a new bass; and Ceres (0.076) may be a baritone. Kepler acknowledged that the celestial choir rarely sang in perfect harmony; perhaps it did so only once, at the moment of creation.[51]

It has been known since antiquity that planetary motions exhibit mutual resonance, and the concept became clearer after it was recognized that the planets orbit a central sun. For example, Earth makes nearly two revolutions about the Sun, while Mars make one complete revolution. A ratio of 2 might suggest that their relative motions correspond to a musical octave. Saturn's orbital period is twenty-nine years, compared with Jupiter's twelve years, a ratio of nearly 5/2, an octave plus a major third. The closest resonance is between Earth and Venus. Eight revolutions of Earth and thirteen revolutions of Venus differ by less than one day. Unfortunately, the ratio of 13/8 does not correspond closely to a note on the musical scale—it is 40 cents, nearly a quarter of a tone, above a major sixth. No exact resonances exist in the Solar System, and attempts to identify musically relevant patterns have met with little success.

More promising is the Titius-Bode Law, named for German astronomers Johann Titius (1729–1796) and Johann Bode (1747–1826). The law hypothesizes that the planets' maximum distance from the Sun—at aphelion—are proportional to 4 + n, where n = 0, 3, 6, 12, 24, 48 .... Thus Mercury's maximum orbital radius is proportional to 4, Venus' to 7, Earth's to 10, Mars' to 16, and so forth. Division by 10 yields the distances in conventional Astronomical Units. Each of the outer planets is nearly twice as far away as its predecessor, like musical octaves. The Titius-Bode Law was a good fit for the planets known at the end of the eighteenth century; moreover, it correctly predicted the orbits of Ceres and Uranus, before they were discovered. But it failed to predict the orbits of Neptune and Pluto.[52]

Music and Modern Esoteric Teachings

Systems of tuning based on integer ratios may have been abandoned. Yet the number 2 retains its significance as the frequency ratio of the octave. It signifies duality, but it also calls to mind the Second Aspect of Deity, Love-Wisdom. The Second Aspect is the form-building aspect of divinity.

Pythagoras is believed to have been a previous incarnation of the Master Koot Hoomi, who now heads the great Second Ray ashram in the Planetary Hierarchy.[53] According to Theosophist Charles Leadbeater, Koot Hoomi is an accomplished musician, owning a musical instrument that can be played as either a piano or a three-manual organ. The instrument, used to communicate with the Gandharvas, creates "combinations of sound never to be heard on the physical plane."[54]

Music and Creation

The great esoteric teachers all spoke of the role of sound in the creation of the universe. Helena Blavatsky spoke of the Buddhist deity Kwan-Yin, "the Divine Voice," who called forth "the Universe out of Chaos and the Seven Elements." Kwan-Yin, she added, dwells in "the "melodious heaven of Sound."[55] Elsewhere, Blavatsky commented that "sound and rhythm are closely related to the four Elements of the Ancients."[56]

Within the Master Koot Hoomi's ashram, several other masters head subsidiary Second Ray ashrams, among them the Tibetan Master Djwhal Khul, who dictated the books of Alice Bailey. Like Pythagoras and Plato, the Tibetan invoked the musical analogy to describe creation: "God created by the power of sound, and the 'music of the spheres' holds all life in being (note that phrase)."[57] Torkom Saraydarian, one of his senior disciples, explained in more detail:

> Sound is the source of all that exists in the Universe. Each atom, each form on any level is composed of sound. The Ageless Wisdom teaches that all communication between created forms is based on sound, sound that is audible and sound that is inaudible. Sound manifests also as light and as energy. . . . [T]he continuity of sound brought into existence seven Cosmic Planes. Each Plane is becoming an octave with seven notes.[58]

We are told that there are seven cosmic planes. The lowest, the cosmic "physical plane," is divided into the seven *systemic planes* of our Solar System. In turn, each systemic plane is divided into seven systemic subplanes, for a total of forty-nine levels of reality in which, to quote *Acts* 17:28, we denizens of the Solar System, live and move and have our being. Saraydarian commented: "These forty-nine planes of Creation are seven octaves, the various combinations of which are the Existence. The Creator is a Composer."[59]

Within the planes and subplanes certain musical correspondences are valid, including resonances among "notes" an octave apart. For example, resonance exists between the fourth etheric subplane—counting from above—and the fourth systemic plane, the buddhic plane. Resonances also exist among the first, *atomic*, subplanes of each plane.

The Logos is both triune and septenary: manifesting in the three aspects of Will, Love-Wisdom, and Active Intelligence, and also manifesting through the seven rays, which "color" the whole of creation.[60] "From the One who is seven goes forth a word. That word reverberates along the line of fiery essence."[61] There are, we understand, seven sacred planets in the solar system.[62] Likewise, Man, made in the image of God, is both triune and septenary. Man consists of monad, soul and personality. He also has seven vehicles, or "bodies," and seven major chakras.

Theosophists Harriette and Homer Curtiss urged that "all occult students form some definite idea of the true meanings and relations of sound, number, color and form, for they stand at the foundation of occult training. . . . For everything in Nature has its voice which speaks in tones so positive that it cannot mislead, if we open our ears to it."[63] The various elements of the human constitution resonate to, and are influenced by, musical tones and chords. The Curtisses commented on the "keynote of the personality":

> That which is called the keynote of any personality is never one musical tone, but a combination of tones, a chord. There are many ways of finding this chord, but the surest way is by listening in the Silence, first asking for guidance, and striving to harmonize yourself with the Divine that is within you, . . . then endeavoring to silence

all other activities of your mind and listen for the answer.[64]

Djwhal Khul's teachings of the subject reveal significant similarities to Plato's discourses on the relationships among body, soul, and spirit.[65] The Tibetan did not require the soul and personality to occupy precise mathematical fractions of the distance between spirit and body, but he did relate them to musical intervals. Moreover, man's spiritual evolution involves the gradual establishment of harmony:

> At first, there is dissonance and discord… and a fight between the Higher and the Lower. But as time progresses, and later with the aid of the Master, harmony of color and tone is produced (a synonymous matter), until eventually you will have the basic note of matter, the major third of the aligned Personality, the dominant fifth of the [soul], followed by the full chord of the Monad or Spirit.[66]

It is interesting that the Tibetan mentioned the major third, which, under Pythagorean, and even just intonation tuning, caused problems because of discord with the tonic–dominant combination. As noted, eventually the problems were solved by tonal temperament. Although temperament was widely condemned as an affront to divine integer ratios, perhaps it helped fight the prevailing "dissonance and discord."

The creative process of sound, which brought the universe into being, is ongoing. Moreover, it is mirrored on a smaller scale in the lives of disciples:

> It might be of value here if students realized that every good speaker is doing a most occult work. A good lecturer (for instance) is one who is doing work that is analogous on a small scale to that done by the solar Logos. . . . He constructs the form, and then when it is constructed, he finishes up by playing the part of the first Person of the Trinity putting his Spirit, vitality and force into it so that it is a vibrant, living manifestation. When a lecturer or speaker of any kind can accomplish that, he can always hold his audience and his audience will always learn from him; they will recognize that which the thought form is intended to convey.[67]

Music, Color, and the Devas

Blavatsky reminded us "that every sound in the visible world awakens its corresponding sound in the invisible realms, and arouses to action some force or other on the Occult side of Nature. Moreover, every sound corresponds to a color and a number."[68]

Theosophist Geoffrey Hodson, who conducted joint research with organist Gordon Kingsley, clairvoyantly saw color and forms on the mental and astral planes when certain pieces of music were played. Some of Hodson's impressions were painted by artists and reproduced along with descriptions in his book *Music Forms.*[69] Hodson observed that "each note, when sounded or sung, produces in addition to wide-ranging effects, a typical form in super physical matter. These forms are colored by the way the sound is produced, and the size of the form is decided by the length of time in which a note is sounded or sung."[70] Kingsley commented on the key and principal theme in Bach's C-Sharp Minor Prelude:

> The minor key always tends to produce a drooping form in contradistinction to the major with its upraised, turreted tendencies. In addition, we have the theme itself as well as the various melodies of the composition repetitions of the theme, all of which move downward. . . . However, the effect upon the hearer is not one of gloom, but of an inward searching, quite in keeping with the mystical character of the key of C sharp minor.[71]

In a similar vein, Austrian esotericist Rudolph Steiner declared:

> Musical creations . . . must be generated anew again and again. They flow onward in the surge and swell of their harmonies and melodies, a reflection of the soul, which in its incarnations must always experience itself anew in the onward-flowing stream of time. Just as the human soul is an evolving entity, so its reflection here on earth is a flowing one.[72]

On another occasion, Steiner observed that "on earth, we can speak and sing only by means of air, and in the air formations of the tone element we have an earthly reflection of a soul-spiritual element. This soul-spiritual element of tone belongs in reality to the super-sensible world, and what lives here in the air is basically the body of tone."[73]

Helena Blavatsky gave teachings on sound and color to special advanced classes. The material was compiled into a "third volume" of her magnum opus, *The Secret Doctrine*. Blavatsky provided tables of correspondences among notes of the scale (which she designated by the *solfège* syllables: Do, Re, Mi, Fa, Sol, La and Si), numbers, colors, planets, the days of the week, metals, and the human principles in Theosophy. Like Nicomachus, she assigned the Moon to the major seventh, but otherwise her planetary correspondences differed from earlier ones.

Blavatsky's correspondences are summarized in Table 6.[74] Based on her planetary correspondences, and to some extent the colors, we also may theorize correspondences with the Seven Rays. These are shown in Table 7.

Table 6. Musical Correspondences from Blavatsky

Number	Degree or Pitch	Color	Planet	Day of the Week	Metal	Human Principle
1	Do	Red	Mars	Tuesday	Iron	Kāma Rūpa, the Seat of Animal Life
2	Re	Orange	Sun (or Vulcan, an undiscovered inter-Mercurial planet)	Sunday	Gold	Prāna, or Life Principle
3	Mi	Yellow	Mercury	Wednesday	Mercury	Buddhi, or Spiritual Soul
4	Fa	Green	Saturn	Saturday	Lead	Lower Manas, or Animal Soul
5	Sol	Blue	Jupiter	Thursday	Tin	Auric Envelope
6	La	Indigo or Dark Blue	Venus	Friday	Copper	Higher Manas, Spiritual Intelligence
7	Si (or Ti)	Violet	The Moon	Monday	Silver	Chhāyā, Shadow or Double

It is possible to design meditations that take advantage of these correspondences in order to be in harmony with the energy of the day and/or the ruling planets of the current ruling astrological sign. For example, a meditation at the full moon of Pisces, ruled by Jupiter, may include a visualization of the color blue and an intonation of the Sacred Word on the note G. The same procedures might be followed for a meditation on Thursday, ruled by the planet Jupiter.

Table 7. Implied Ray Correspondences for Musical Pitches

Degree or Pitch	Ray
Do	Ray 6
Re	Ray 1
Mi	Ray 4
Fa	Ray 3

Degree or Pitch	Ray
Sol	Ray 2
La	Ray 5
Si (or Ti)	Ray 7

The effectiveness of chanting depends on the purity of the vowels sounded. In this regard, an understanding of the formants—the patterns of harmonics in the voice—may be especially important. The reader is referred to the discussion of formants earlier in this article.

The correspondences presented above are open to differences of opinion and interpretation. The subject of color, and doubtless music as well, is subject to occult "blinds," or a substitution of imprecise or incorrect knowledge to protect the inexperienced student, or perhaps to make him or her think things out in an effort to resolve the apparent contradictions. For example, in Tables 6 and 7, Blavatsky equates the note G with the color blue. In the following passage on the passing out of the Sixth Ray and the coming in of the Seventh Ray, however, the Tibetan Master seems to equate the note G with violet and the Seventh Ray of Ceremonial Magic. He also associates the color blue with the Sixth Ray of Devotion:

> The blue ray of devotion passes now into the violet of what we term the ceremonial ray. What do these words mean? Simply that the great Musician of the universe is moving the keys, is sounding another note and thus bringing in another turn of the wheel, and swinging into the arc of manifestation the ray of violet, the great note G.[75]

In his clairvoyant investigations, Hodson observed that the devas were involved with the performance of music and the generation of music forms. Esotericist and composer Cyril Scott wrote that certain composers, such as Richard Wagner, César Franck, and Alexander Scriabin were particularly influenced by the deva evolution. According to Scott, music in the future will be used "to bring people into yet closer touch with the Devas," and that people "will be enabled to partake of the benefic influence of these beings while attending concerts at which by the appropriate type of sound they have been invoked."[76]

Much more can be said about music, color, the rays, and the deva evolution. These topics will be explored in detail in future articles. The present comments are included to indicate the direction the larger research program is heading.

Music and Service

The composition and performance of music can be great acts of service. Music has the potential to bring about transformative effects at physical, emotional, mental and spiritual levels. Music's ability to heal maladies of body and soul was known in many ancient cultures and, as noted, is said to have been employed by Pythagoras. Today, music therapy is an important element of complementary medicine. The long tradition of sacred music affirms trust in its efficacy in raising human consciousness.

Virtuoso performance can, of course, express music's highest excellence. But to sing in a choir or play in an orchestra or band—even of less obvious quality—is group service, expressing harmony in multiple senses of the word. The Tibetan Master emphasized the beneficial effects of music:

> The soul on its tiny scale can create "the new man" by the power also of sound, and a musical rhythm can usefully be imposed upon the personality life by the disciple. . . . Let love and light and music enter more definitely into your daily life. Spurn not this practical suggestion, but give your mind the opportunity, through the massed sound of music, to break down the personality-imposed barriers between the free flow of soul life and you.[77]

Moreover, he looked forward to a time when music will play a more specific role: "In time to come the value of the combination of music, chanting, and rhythmic movement will be comprehended, and it will be utilized for the achieving of certain results. Groups of people will gather together to study the creative effects, or the purificatory efficacy of ordered sound joined to movement and unity."[78] The potential for human service is impressive. In the future, we are told, man will rediscover some of the powers of music to alter matter. One of these powers

> will grow out of the study of sound and the effect of sound and will put into man's hands a tremendous instrument in the world of creation. Through the use of sound the scientist of the future will bring about his results; through sound, a new field of discovery will open up; the sound which every form in all kingdoms of nature gives forth will be studied and known and changes will be brought about and new forms developed through its medium. One hint only may I give here and that is, that the release of energy in the atom is linked to this new coming science of sound.[79]

Sound can also be destructive in intent, helping to break up forms that have outlived their usefulness. Sound plays an all-important role when individuals with Fourth Ray souls approach the fourth initiation: "When the egoic note is sounded in harmony with other egos, the result is the shattering of the causal body, dissociation from the lower and the attainment of perfection."[80]

Conclusions

This article has examined music theory and practice, from antiquity to the present, to identify the mathematical underpinnings of musical harmony and to explore their esoteric significance. Vibration and rhythm are built into the very fabric of reality. For its part, mathematics has succeeded in modeling the physical universe to a degree that amazes scientists. For example, Hungarian mathematical physicist Eugene Wigner exclaimed: "It is difficult to avoid the impression that a miracle confronts us here."[81] In many areas mathematics transcends the physical, and for many people, including many mathematicians, it merges into mysticism. Mathematics may be considered a branch of esotericism in its own right.

Not surprisingly, the relationship between music and mathematics, and their mutual relationship with esotericism, are subjects of profound importance. To study these subjects is of intrinsic value, as well as enhancing the experience of musical composition, performance and audition.

As the Pythagoreans—a term that should include the Samian's predecessors and successors—were well aware, music and number are closely correlated. To a lesser but still important extent musical intervals have geometric correlations. While the vision of usable musical scales defined solely by integer ratios was never realized, many of the mathematical correlations survive.

The most important number in music theory is obviously 2; each octave in the ascending musical scale represents a doubling of frequency. Within the octave, frequency ratios based on 2, 3, and possibly 5 have been abandoned, but in their place we have fractional powers of 2 and *cents*, their linearized logarithmic transforms. "Two," the *duad*, is the first emanation from the primeval Unity; it is the feminine principle which, together with the primeval masculine, makes possible the emergence of complexity and infinite potential. Importantly, the Second Aspect of Deity is the form-building aspect, perhaps mirroring the role of sound in creation.

The natural numbers: 1, 2, 3, 4... define the harmonic series. In addition to the duad, two numbers of great importance in music are 7 and 12: the number of notes in the diatonic and chromatic scales, respectively. Seven, or $3 + 4$, and the largest prime less than 10, has always been considered a number of major esoteric significance. The *heptad*, "sacred number," or "virgin number," was encoded in the seven days of the week, the seven Christian sacraments, and the seven colors of the spectrum. Modern esoteric teachings have added the seven rays, the seven sacred planets, the seven planes, and the seven major chakras. Twelve, or 3×4, is also of major esoteric significance. The *duodecad* was encoded in the twelve tribes of Israel, the twelve disciples of Christ, the twelve months of the year, the twelve signs of the zodiac, and the twelve hours of the day or night. Esoteric teachings have added the twelve Creative Hierarchies.

Defining the major third in the musical scale was challenging and became a critical factor leading to the abandonment of integer ratios and the acceptance of temperament. Yet, even in the modern musical scale, the fourth and fifth remain close to the Pythagorean frequency ratios of 4/3 and 3/2, respectively. The chord consisting of the root, major third and fifth is one of the most common in musical composition, and modern esoteric teachings relate the major third to the integrated personality and the fifth to the soul.

Whether Pythagoras could hear the harmony of the spheres, we shall never know. But we do know that sound waves cannot propagate through interplanetary space, so any such harmony must be sought either in mathematical relationships or in extra-physical realms of awareness. As Kepler's work demonstrated, interest in the harmony of the spheres survived the transition from a geocentric to a heliocentric understanding of the Solar System. The notion of planets of a "celestial choir," whose vocal range depended on orbital eccentricity, was an evocative concept—even though it raises many questions. Like Isaac Newton, born twelve years after Kepler's death, Kepler's scientific investigations were overlaid with powerful insights into higher reality.

The spoken word, the sacred AUM, or various forms of celestial music have played critical roles in the creation myths of the world religions. Sound had the ability to produce order out of chaos, and, as the Pythagoreans declared, certain musical intervals had special potency in that regard. Modern science has demonstrated sound's organizing power by Ernst Chladni's sand patterns on a vibrating plate and by the acoustic levitation and manipulation of small objects.[82] Sound waves may have played a role in the formation of galaxies in the early universe. In the phenomenon of sonoluminescence, sound waves can cause bubbles of liquid to implode and emit bursts of light.

Music has had an enormous impact on human civilization. War cries, work songs, lamentations, hymns of joy or praise, lullabies, music in entertainment, sacred music: the list is almost endless. Music has a profound influence on the human constitution, from the physical, to the emotional and mental levels, and beyond.

Cyril Scott wrote that in the future, the unity of color and sound will be realized, along with its healing and stimulating effects.[83] Synesthesia is a rare condition today, but it draws attention to the relationship between music and color, supporting esoteric teachings on the convergence of sound and color on the higher planes. It may also promise an evolutionary future in which much larger numbers of people are so-gifted.

The work of Geoffrey Hodson and others has drawn attention to music's power to attract devic beings, and much of the power of sacred ritual no doubt lies in devic participation. Esoteric teachings alert us to the role that senior disciples in the Fourth Ray Ashram may have played, throughout history, in the composition and performance of music intended to stimulate the unfoldment of human consciousness. We understand that the Fourth Ray will begin to come into manifestation in 2025, and the joint efforts of the Planetary Hierarchy, the Deva Evolution, and enlightened humanity will result in a new golden age of music and the arts.

This article has touched on just a few aspects of the esoteric significance of music. Much more remains to be done, to understand our musical heritage, to encourage relevant discipleship work, and to prepare for the time when humankind is entrusted with "a tremendous instrument in the world of creation."

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1 Online at: http://commons.wikimedia.org/wiki/File:Jan_van_Eyck_-_The_Ghent_Altarpiece_-_Angels_Playing_Music_-_WGA07644.jpg. Image from Web Gallery of Art. This is a photographic reproduction of a two-dimensional, public domain work of art. The work of art itself is in the public domain for the following reason:

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2 James Owen, "Bone Flute Is Oldest Instrument, Study Says," *National Geographic News* (June 24, 2009). Online: http://news.nationalgeographic.com/news/2009/06/090624-bone-flute-oldest-instrument.html. (Last accessed Jan. 27, 2015.)

3 *Genesis* 1:1,3, KJV. Emphasis added.

4 Job 38:7, KJV.

5 See for example Ernest G. McClain, *The Myth of Invariance* (York Beach, ME, Nicholas-Hays, 1976); R. A. Schwaller de Lubicz, *The Temple of Man* (Rochester, VT: Inner Traditions, 1981.)

6 Alice A. Bailey, *Initiation, Human and Solar* (New York: Lucis: 1922), 60.

7 See for example: Celeste Jamerson, *Franz Liszt and the Seven Rays*, *The Esoteric Quarterly* (Spring 2014), 15; Donna M. Brown, *Richard Wagner: An Esoteric Perspective*, *The Esoteric Quarterly* (Winter 2015), 55.

8 The product of frequency and wavelength is the speed of sound, which increases with temperature. As the temperature increases, a particular note is associated with a progressively shorter wavelength, and musical instruments have to be retuned when moved from one performance environment to another. The wavelengths quoted here correspond to typical room temperature.

9 A standard piano keyboard runs from A0 to C8; thus A4 is the fifth A from the bottom of the range.

10 In scientific pitch, the frequency of every C in the musical scale is an integral power of 2.

11 Ten octaves correspond precisely to a ratio of 2^{10}, or 1,024, in frequency.

12 The frequency of a vibrating string also depends on the string's tension and its density (mass per unit length).

13 Online: http://commons.wikimedia.org/wiki/File:Spectrogram_-iua-.png. This file is licensed under the Creative Commons Attribution 2.0 Generic license. Created by en.User:ish ishwar in 2005. (Last accessed March 21, 2015.)

14 "Harmonic intervals," in this context, refer to the simultaneous sounding of notes, without any implication that the notes belong to the harmonic series, or that they necessary sound "harmonious."

15 An exception to this is the harmonic minor scale, which contains an augmented second (one and a half steps) between the 6th and 7th scale degrees, or notes of the scale. In the harmonic A-minor scale, these notes would be F and G#. Further explanation of scale degrees is given at the end of this section.

16 The reader may verify that multiplying 1.059463... (the decimal equivalent of $2^{1/12}$) by itself twelve times produces the value 2. The 1/12th power of 2, $2^{1/12}$, can also be written as $\sqrt[12]{2}$, the 12th root of 2.

17 The general rule for multiplying numbers raised to powers is $x^a \times x^b = x^{(a+b)}$.

18 The transformation: $1200\log_2 r$, where r is the frequency ratio, was proposed in the 1830s by Frenchmen Gaspard de Prony and Robert Bosanquet.

19 To understand why an interval extending over two notes is called a "third," think of fence posts. Three posts are needed to support two spans of fence.

20 The natural minor scale is also referred to as the Aeolian mode. The variants have different semitone patterns.

21 When remote keys and harmonies are accessed, the use of the double flat and the double sharp also occur from time to time.

22 *Liber Usualis* (Tournai, Belgium: Desclee, 1961), 1,504. Each of the first six phrases of the hymn begins on a higher note of the scale, and the second through sixth phrase begin with the syllables mi, fa, sol and la. A recording of the hymn and accompanying score can be found at https://www.youtube.com/watch?v=7-WtmOniiRw. (Last accessed March 30, 2015.)

23 Pianos are designed so that strings are struck one-seventh the way along their length to suppress the discordant seventh harmonic.

24 The figure is a modified form of an image provided in http://en.wikipedia.org/wiki/Circle_of fifths.

25 The root characterizes the chord but is not necessarily the lowest of the several pitches. The situation in which one or more pitches lie below the root is referred to as *inversion*.

26 Franchino Gaffurio, *Theoretica Musica*, 1492.

27 This instrument, with a string length of one meter (39.4 inches), was constructed by one of the authors.

28 Primes are used here simply to denote octaves above an arbitrary C on the musical scale. C' is one octave, and C" two octaves, above the reference C. This notation should not be confused

with the archaic Helmholtz pitch system, in which primes denoted specific octaves.

29 The more practical solution would be to take the geometric mean of the ratios for F and G: $\sqrt{}(4/3 \times 3/2) = 1.414...$, or precisely 500 cents. But this result was unacceptable to the Pythagoreans because it could not be expressed as a ratio of integers.

30 The authors are indebted to a reviewer for drawing attention to this usage by modern composers.

31 That the comma occurred after *twelve* fifths and *seven* octaves seems to resonate with the tradition that there are seven notes in the diatonic and twelve in the chromatic scale.

32 See for example: Stuart Isacoff, *Temperament* (New York: Alfred Knopf, 2001), 97-118.

33 Some authorities apply the term "just intonation" to all tuning systems based on integer ratios. They refer to systems in which 5 is the largest admissible prime number as "five-limit tuning." Correspondingly, Pythagorean tuning, where 3 is the largest admissible prime number, becomes "three-limit tuning."

34 Isacoff, *Temperament*, 182.

35 McClain, *The Myth of Invariance*, 11.

36 Dane Rudhyar, *The Magic of Tone and the Art of Music* (Boulder, CO: Shambhala, 1982), 167-68.

37 The same is true of the ratios: 1, 4/3, 3/2, 2. The arithmetic mean of any two numbers a and b is defined as $(a + b)/2$. The harmonic mean is $2ab/(a + b)$, and the geometric mean is $\sqrt{}(ab)$.

38 For an extensive discussion of the arithmetic, harmonic and geometric means of these numbers, and their geometric and numerological significance, see David Fideler, *Jesus Christ: Sun of God* (Wheaton, IL: Quest, 1993), 87-101, 220-223.

39 Christianity preserved the sacredness of the heptad in the seven sacraments, seven cardinal virtues, and seven deadly sins. Esoteric teachings preserve it in several correspondences to be discussed shortly.

40 Plato, *Timaeus* 31C-32C. Desmond Lee (ed.), *Timaeus and Critias* (London: Penguin Classics, 1965), 44.

41 Ibid., 31C.

42 The formula is interpreted as $F/A = A/W$, $A/W = W/E$. Solving for W and A in terms of A and F, we obtain $W = (E^2F)^{1/3}$, $A = (EF^2)^{1/3}$. With $E = 1$, $F = 2$, the end-result is $W = 2^{1/3}$, $A = 2^{2/3}$.

43 "Pythagoras," *Stanford Encyclopedia of Philosophy*. Online: http://plato.stanford.edu/entries/pythagoras/. (Last accessed March 14, 2015.)

44 Fabre d'Olivet, *The Secret Lore of Music*, trans. Joscelyn Godwin from the 1928 Edition of Jean Pinasseau (Rochester, Vermont: Inner Traditions, 1987).

45 The Fibonacci ("son of Bonnaci") series, named for Leonardo Bonacci (c.1170–c.1250), is a series of integers in which each term is the sum of the previous two. The Fibonacci series has strong connections with the spiral configurations found in plant and animal species, and with the Golden Rectangle, encoded in the architecture of many ancient buildings.

46 Flora R. Levin, *The Manual of Harmonics of Nicomachus the Pythagorean* (Grand Rapids, MI: Phanes), 1994, 47-53.

47 Henry C. Agrippa, *Three Books of Occult Philosophy* (trans. J. Freake), reprint (Woodbury, MN: Llewellyn), 1651/2006, book 2, ch 26, 339.

48 Article on "Robert Fludd." Online: http://www.encyclopedia.com/topic/Robert_Fludd.aspx. It will be noted that the string positions of the second octave are incorrect. See the comment in Kenneth S. Guthrie, *The Pythagorean Sourcebook and Library* (Grand Rapids, MI: Phanes), 1987, 326.

49 The eccentricity of an ellipse is defined as $\sqrt{}(1 – b^2/a^2)$, where a and b are the major and minor axes, respectively. The planetary eccentricities listed in Figure 11 are based on modern observations.

50 The alto lines in many four-part choral works are constrained in range and, in the view of some choristers, tuneless. Kepler evidently had a strong opinion in that regard, assigning to the alto section Venus who could sing only a single note.

51 Johannes Kepler, *Harmonices Mundi* (trans.: Charles G. Watts), 1619/1939, §§5-6, 34-43.

52 The largest errors, on the order of 5 percent, occur for Mars and Saturn. The law predicts Jupiter's orbit almost exactly. But Pluto's orbit is only one-half of the predicted distance.

53 Charles W. Leadbeater, *The Masters and the Path* (Adyar, India: Theosophical Publishing House, 1953), 272. See also Manly P. Hall, *The Secret Teachings of All Ages* (Los Angeles: Philosophical Research Society), 1928, 65.

54 Leadbeater, *The Masters and the Path*, 34-35.

55 Helena P. Blavatsky, *The Secret Doctrine*, I (Los Angeles: Theosophical University Press, 1888), 137.

56 Ibid., 307.

[57] Alice A. Bailey, *Discipleship in the New Age* II (New York: Lucis, 1955), 699-700. Parenthesis in original.

[58] Torkom Saraydarian, *The Creative Sound* (Cave Creek, AZ: TSG Publishing Foundation, 1999), 33.

[59] Ibid., 116.

[60] The mathematical result mentioned earlier, that there are seven possible combinations of three elements, can be viewed as a metaphor for the progression of divine manifestation from the trinity to the seven rays.

[61] Alice Bailey, *Esoteric Psychology*, I (New York: Lucis), 1962, 63

[62] Bailey, *Initiation, Human and Solar*, 2.

[63] Harriette A. & F. Homer Curtiss, *The Voice of Isis* 2/e (Los Angeles: Curtiss Book Co., 1914), 356. The Curtisses founded the Order of Christian Mystics in 1908.

[64] Ibid., 358.

[65] John F. Nash, "Plato: A Forerunner," *The Beacon* (July/August 2004), 18-24. There is speculation that the Master Djwhal Khul might be the reincarnation of Plato.

[66] Alice A. Bailey, *Letters on Occult Meditation* (New York: Lucis, 1922), 62. Parenthesis in original.

[67] Alice A. Bailey, *A Treatise on Cosmic Fire* (New York: Lucis, 1925), 979.

[68] Helena P. Blavatsky, *The Secret Doctrine*, III (Benares, India: Theosophical Publishing Society, 1897), 451.

[69] Hodson, Geoffrey. *Music Forms: Superphysical Effects of Music Clairvoyantly Observed* (Adyar, India: Theosophical Publishing House, 1976), vii.

[70] Ibid., 19.

[71] Ibid., 21-22.

[72] Rudolf Steiner, lecture, Nov. 12, 1906. Rudolf Steiner Archives. Online: http://wn.rsarchive.org/Lectures/GA283/English/AP1983/19061112p01.html. (Last accessed March 17, 2015.)

[73] Rudolf Steiner, lecture, Dec. 2, 1922. Rudolf Steiner Archives. Online: http://wn.rsarchive.org/Lectures/GA283/English/AP1983/19221202p01.html. (Last accessed March 17, 2015.)

[74] Helena P. Blavatsky, *The Esoteric Writings of Helena Petrovna Blavatsky: A Synthesis of Science, Philosophy and Religion* (Wheaton, IL: Theosophical Publishing House, 1980), 379 and Diagram II after 360. "Originally published in 1907 as volume 3, Occultism, of The Secret Doctrine."

[75] Bailey, *Esoteric Psychology*, I (New York: Lucis, 1962), 121-122.

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